G finite group.

$GL_n = GL(n, \mathbb{C})$

V = vector space $/\mathbb{C}$.

① Matrix rep'n.

$R: G \longrightarrow GL_n$ group homomorphism

$\chi_R : G \to \mathbb{C}$ character function.

$$\chi_R(g) = Tr(R(g))$$

② G operates on V linearly

or V is a G-rep'n.

$G \times V \to V$

$(g, v) \mapsto g.v$.

2.1. $(g_1 g_2) \cdot v = g_1 (g_2 v)$ $g_1, g_2 \in G$, $v \in V$.

2.2. $e \cdot v = v$

2.3. $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$ $v_1, v_2 \in V$

$g(c \cdot v) = c \cdot (g \cdot v)$ $c \in \mathbb{C}$, $v \in V$

③ $\rho: G \to GL(V)$

$= \{$ Invertible linear transformations $V \to V\}$.

$\boxed{② \Leftrightarrow ③.}$

② ⇒ ③, $g \mapsto \left(\begin{array}{l} m_g : V \to V \\ \quad v \mapsto g \cdot v \end{array} \right)$

③ ⇒ ② : $g \cdot v = \rho(g)(v)$

① ⇐ ②, ③ by choice of basis.

If $B = \{v_1 \cdots v_n\}$ is a basis of V,

then $R_B(g)$ is defined to be the matrix of linear transformation

$m_g : V \to V$ under basis B.
$\quad v \mapsto g \cdot v$

$$g(v_1, \dots, v_n) = (v_1, \dots, v_n) \cdot R_B(g)$$

$$R(g) = \begin{bmatrix} g_{11} & \cdots & \cdots \\ g_{21} & \cdots & \cdots \\ g_{31} & \cdots & \cdots \\ \vdots & & \\ g_{n1} & \cdots & \cdots \end{bmatrix}$$

$$g v_1 = (v_1 \cdots v_n) \cdot \begin{pmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{n1} \end{pmatrix}$$

$$= \sum_{k=1}^{n} g_{k1} \cdot v_k$$

$$R_B : G \to GL(n, \mathbb{C})$$

① $\Rightarrow$ ②, ③. by choosing $V = \mathbb{C}^n$

and $g \cdot v = R(g) \cdot v$.

Different choice of basis: $C = \{w_1, \dots w_n\}$.

$$(w_1 \cdots w_n) = (v_1 \cdots v_n) \cdot P$$

P $n \times n$ invertible.

is $(P_{B \leftarrow C})$ change of basis matrix.

$$g(w_1 \cdots w_n) = (w_1 \cdots w_n) \cdot R_C(g)$$

$$g(v_1 \cdots v_n) P = (v_1 \cdots v_n) \cdot P \cdot R_C(g).$$

So $g(v_1 \cdots v_n) = (v_1 \cdots v_n) \cdot P R_C(g) P^{-1}$

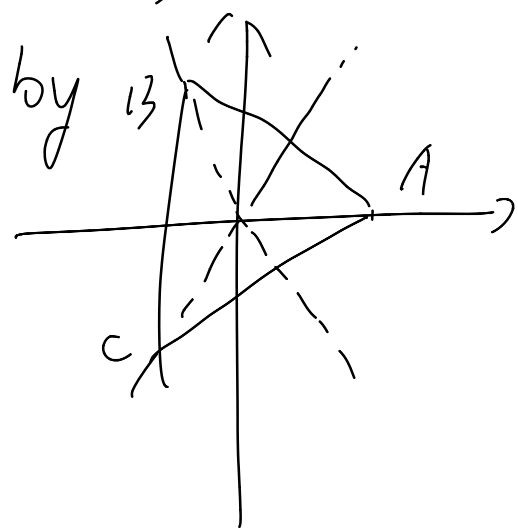
so $\boxed{R_B(g) = P R_C(g) \cdot P^{-1}.}$

Prop: χ_ρ is well-defined

since $(\mathrm{tr}\, R_B(g))$ does not depend on choice of basis.

Prop: If $g_1 \sim g_2$ are in the same conjugacy class, then
$$\chi_\rho(g_1) = \chi_\rho(g_2).$$

Ex: $S_3 = D_3 \xrightarrow{R} GL_2(\mathbb{C})$.



x rotation by $120°$.

y reflection with respect to x-axis.

$$R(x) = \begin{bmatrix} \cos 120°, & -\sin 120° \\ \sin 120°, & \cos 120° \end{bmatrix}$$

$$R(y) = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}.$$

$\chi_R(x) = -1$.

$\chi_R(y) = 0$.

V_1, V_2 rep'ns of G.

$f: V_1 \to V_2$ linear isomorphism.

If $f(g \cdot v) = g \cdot f(v)$, then f is called an isomorphism of G-rep'ns.

Prop: ρ_1, ρ_2 are isomorphic, and choose $B = \{v_1 \dots v_n\}$ a basis of V_1. and $B' = \{f(v_1) \dots f(v_n)\}$ a basis of V_2. then $R_B = R_{B'}$. Hence $\chi_{\rho_1} = \chi_{\rho_2}$.

Invariant subspace.

Defn W is invariant subspace if

$$g \cdot w \in W, \ \forall g \in G, w \in W.$$

Prop: $gW = W$.

Trick: averaging under G-operation.

$v \in V$,

$$w = \frac{1}{|G|} \sum_{g \in G} g \cdot v.$$

then $gw = w \quad \forall g \in G$.

$\mathbb{C}w$ is an invariant subspace.

Direct sum:

(External) W_1, W_2 are two vectorspaces,

$$W_1 \underset{E}{\oplus} W_2 = W_1 \times W_2 = \{(w_1, w_2) \mid w_i \in W_i\}.$$

① addition $(w_1, w_2) + (v_1, v_2) = (w_1+v_1, w_2+v_2)$

② scalar multiplication $c \cdot (v_1, v_2) = (cv_1, cv_2)$

(Internal) $W_1, W_2 \subset V$ subspaces.

① $W_1 \cap W_2 = \{0\}$

② $W_1 + W_2 = \{w_1 + w_2 \mid w_i \in W_i\}$
$= V$.

then $V = W_1 \underset{I}{\oplus} W_2$.

Prop: $V = W_1 \underset{I}{\oplus} W_2$ then

$$V \cong W_1 \underset{E}{\oplus} W_2.$$

Prop: $V = W_1 \underset{E}{\oplus} W_2$, then $V = \widetilde{W_1} \underset{I}{\oplus} \widetilde{W_2}$

$\widetilde{W_1} = \{(w_1, 0) \mid w_1 \in W_1\}$, $\widetilde{W_2} = \{(0, w_2) \mid w_2 \in W_2\}$.

If W_1, W_2 are both G-invariant.

then V is a direct sum of

W_1, W_2 as G-repn.

Ex: S_3 operates on $\mathbb{C}^3$ by

$\sigma \cdot e_i = e_{\sigma(i)}$. (e_1, e_2, e_3) standard basis

$W_1 = \frac{1}{3}(e_1 + e_2 + e_3)$

$W_2 = \{ a e_1 + b e_2 + c e_3 \mid a+b+c=0 \}$.

then $\mathbb{C}^3 = W_1 \oplus W_2$.

Irreducible repn.

Defn: If V has only trivial G-invariant subspaces $\{0\}$, V, then V is called irreducible G-repn.

Ex: 1-dim'l rep'ns.

Unitary rep'n.

If V is a G-rep'n. and

$\langle , \rangle$ is a positive definite Hermitian form

and $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$, $v, w \in V$, then we say

V is unitary rep'n.

Prop: If V is unitary, then $\forall$ W G-invariant. $W^\perp$ is also G-invariant.

Prop: V is unitary, then V is the direct sum of irreducible rep'ns

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

Pf: If V is not irreducible.

V has G-invariant subspace

W, $\dim W > 0$, $\dim W < V$.

so $V = W \oplus W^\perp$

and $\dim W^\perp < \dim V$.

Use induction on dim.

(Maschke's Thm).

Every G-rep'n V is the direct sum

of irreducible subspaces.

Pf: Every V is unitary rep'n.

$\langle \, , \rangle$ is a positive definite Hermitian

form.

$$\langle v, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle.$$

then $\langle \, , \, \rangle_G$ is preserved by G.

$$\langle gv, gw \rangle_G = \langle v, w \rangle_G .$$

Orthogonal relations. $G = \coprod_{i=1}^{r} C_i$.

C_i conjugacy classes

$\mathcal{H}$ space of class function. $G \to \mathbb{C}$

$\dim \mathcal{H} = r$.

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g)$$

$$= \frac{1}{|G|} \sum_{i=1}^{\ell} c_i \overline{\varphi(g_i)} \psi(g_i)$$

$c_i = |C_i|$, $g_i \in C_i$.

Prop: ρ_1, ρ_2 two irreducible repn's with characters χ_1, χ_2.

① $\langle \chi_1, \chi_2 \rangle = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ 1 & \text{if } \rho_1 \cong \rho_2 \end{cases}$

② The number of isomorphism classes of irreducible repn's $= r$.

Denote by $\chi_1 \cdots \chi_r$ the corresponding characters.

① $\Rightarrow$ $\chi_1 \cdots \chi_r$ linearly independent.

$+$ ② $\Rightarrow$ $\chi_1 \cdots \chi_r$ is a basis of $\mathcal{H}$.

$\Downarrow$

g_1, g_2 are in the same conjugacy class if and only if $\chi(g_1) = \chi(g_2)$ for all character χ.

③ χ is irreducible iff $\langle \chi, \chi \rangle = 1$

④ $\rho = n_1\rho_1 \oplus n_2\rho_2 \oplus \cdots \oplus n_r\rho_r$

$\chi = n_1\chi_1 + \cdots + n_r\chi_r$.

$n_i = \langle \chi, \chi_i \rangle$.

⑤ $\rho \cong \rho'$ iff $\chi_\rho = \chi_{\rho'}$.

" $\Rightarrow$ "

" $\Leftarrow$ " $\langle \chi_\rho, \chi_i \rangle = n_i$

(b) $\sum (d_i)^2 = |G|$.

$$\begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_r \\ \hline \chi_1 & & & & \\ \vdots & & & & \\ \chi_r & & & & \end{array} \qquad X.$$

Matrix X. $\begin{bmatrix} \text{———} \ \chi_1 \ \text{———} \\ \\ \\ \end{bmatrix}$

$$\frac{1}{|G|} \cdot \overline{X} \cdot \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_r \end{bmatrix} \cdot X^T = id$$

$$X^T \cdot \overline{X} \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_r \end{bmatrix} = \begin{bmatrix} |G| & & \\ & \ddots & \\ & & |G| \end{bmatrix}.$$

second orthogonal relations:

$$\underbrace{X^T \cdot \overline{X}}_{\downarrow} \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_r \end{bmatrix} = \begin{bmatrix} |G| & & \\ & \ddots & \\ & & |G| \end{bmatrix}$$

$$\underbrace{\sum_{k=1}^{r} \chi_k(g_i) \overline{\chi_k(g_j)}}_{\text{orthogonal relations of column vectors.}} = \begin{cases} \frac{|G|}{c_i} & i=j \\ 0 & i \neq j \end{cases}$$

First column. $\Rightarrow \sum_{i=1}^{r} (d_i)^2 = |G|$

G abelian group. $|G| = n$.

$r = n$, $\sum_{i=1}^{n} (d_i)^2 = n$.

$d_i = 1$.

Prop: G is abelian iff all the irreducible rep'n's are 1 dim'l.

special example $\mathbb{Z}/3\mathbb{Z}$.

	e	a	a^2
χ_1	1	1	1
χ_2	1	w	w^2
χ_3	1	w^2	w

$w = e^{\frac{2\pi i}{3}}$

The product of any two characters is still a character.

Prop: If χ_0 is a 1-dim'l character, χ is a character, then $\chi_0 \cdot \chi$ is a character of G.

Pf: $\rho : G \to GL(V)$

$\rho_0 : G \to \mathbb{C}^*$

Check $\rho_0 \cdot \rho(g) = \rho_0(g) \cdot \rho(g)$ is a group homomorphism and $tr(\rho_0 \rho)(g) = \chi_0(g) \cdot \chi(g)$

Prop: χ is irreducible iff $\chi_0 \chi$ is irreducible.

pf: $\langle \chi_0\chi, \chi_0\chi \rangle$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_0(g)\,\chi(g)} \cdot \chi_0(g) \cdot \chi(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\overline{\chi_0(g)} \cdot \chi_0(g)}_{=1 \text{ because } \rho_0 : G \to U(1)} \cdot \overline{\chi(g)} \cdot \chi(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)}\, \chi(g)$$

$$= \langle \chi, \chi \rangle .$$

How to find character table for S_4.

S_4. 5 conjugacy classes.

	1	6	8	3	6
	(1)	(12)	(123)	$(12)(34)$	(1234)
χ_1	1	1	1	1	1

↑
trivial repn.

$\rho_{perm} : S_4 \to GL(4)$.

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\chi_{perm}(\sigma) = \#\{ i \in \{1 \cdots n\} \mid \sigma(i) = i \}.$$

	1	6	8	3	6
	(1)	(12)	(123)	$(12)(34)$	(1234)
χ_{perm}	4	2	1	0	0

$$\langle \chi_{perm}, \chi_{perm} \rangle = \frac{1}{24}(1\cdot 4\cdot 4 + 6\cdot 2\cdot 2 + 8\cdot 1\cdot 1 + 3\cdot 0 + 6\cdot 0)$$

$$= 2 \quad = 1^2 + 1^2$$

So χ_{perm} = two irreducible repn's.

$\langle \chi_{perm}, \chi_1 \rangle = 1$.

So $\chi_{perm} = \chi_1 + \chi_2$.

	1	6	8	3	6
	(1)	(12)	(123)	$(12)(34)$	(1234)
χ_1	1	1	1	1	1
$\chi_2 = \chi_{perm} - \chi_1$	3	1	0	-1	-1.

def ρ_{perm} = sign of σ. we get χ_3.

	1	6	8	3	6
	(1)	(12)	(123)	$(12)(34)$	(1234)
χ_3	1	-1	1	1	-1.

$\chi_3 \cdot \chi_2$ 3 -1 0 -1 1

	1	6	8	3	6
	(1)	(12)	(123)	$(12)(34)$	(1234)
χ_1	1	1	1	1	1
χ_2	3	1	0	-1	-1
χ_3	1	-1	1	1	-1.
$\chi_3 \cdot \chi_2 = \chi_4$	3	-1	0	-1	1
χ_5	d_5	a	b	c	d.

second orthogonal relations.

$$(d_5)^2 + 1^2 + 3^2 + 1^2 + 3^2 = 24 \Rightarrow d_5 = 2.$$

$$1\cdot 1 + 3\cdot 1 + 1\cdot(-1) + 3\cdot(-1) + a\cdot d_5 = 0 \Rightarrow$$

$$a = 0,\quad b = -1,\quad c = 2,\quad d = 0$$

$\ker \rho = \{ g \mid \chi(g) = \chi(1) \}$ Homework.
$= \ker \chi$

We can read out normal subgroups of S_4 from the table.

$\ker \chi_1 = S_4$

$\ker \chi_2 = \{(1)\}$

$$\ker \chi_3 = \{ (1), \begin{matrix} (132) & (143) & (142) \\ (123), & (134), & (124), \\ (234), & (243), & \\ (12)(34), & (13)(24), & (14)(23) \end{matrix} \}$$

$= A_4$.

$\ker \chi_4 = \{(1)\}$

$\ker \chi_5 = \{ (1), (12)(34), (14)(23), (13)(24) \}$.

Actually all the normal subgroups arises this way.

Any N a normal subgroup of G can be written as $N = \ker \chi$ for some character χ

(This will be proved after <u>regular rep'n</u>)

Permutation representation.

Ex: $S_n \to GL(n)$ is induced by

S_n-operation on $\mathbb{C}^n$,

$\sigma \cdot e_i = e_{\sigma(i)}$.

More generally, we consider

G operation on a set S

with $|S| = n$.

$G \times S \to S$.

then $G \to \{\text{Bijections } S \to S\} \cong S_n$.

So we get a rep'n,

$G \to S_n \to GL(n)$.

Another interpretation: (More intrinsic)

$\mathbb{C}^S$ = vector space with elements
$\sum_{i=1}^{n} a_i s_i \qquad a_i \in \mathbb{C}, \quad s_i \in S.$
(formal linear combinations).

$\mathbb{C}^S$ has basis $s_1 \ldots s_n$

G operates on $\mathbb{C}^S$ linearly by
$$g \cdot \left(\sum_{i=1}^{n} a_i s_i\right) = \sum_{i=1}^{n} a_i \, g \cdot s_i$$

So we have a G rep'n $\mathbb{C}^S$, ρ.

Prop: $\chi_\rho(g) = \#\{s \in S \mid g \cdot s = s\}$.

Pf: The same as the midterm question.

regular rep'n.

G operates on G itself by

$$g * h = g \cdot h.$$

The induced G-rep'n $\mathbb{C}^G$ is called regular rep'n. ρ_{reg}

Prop: $$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e. \end{cases}$$

pf: $$\{h \mid g \cdot h = h\} = \begin{cases} G & \text{if } g = e \\ \phi & \text{if } g \neq e. \end{cases}$$

Cor: Let $\rho_1 \dots \rho_r$ be all the irreducible repn's.

$$\rho_{reg} = \sum_{i=1}^{r} n_i \rho_i, \quad n_i = \dim \rho_i$$

pf: $$\langle \chi_{reg}, \rho_i \rangle = \frac{1}{|G|} \cdot \sum_{g \in G} \chi_{reg}(g) \cdot \chi_i(g)$$

$$= \frac{1}{|G|} \cdot \underbrace{\chi_{reg}(e)}_{|G|} \cdot \underbrace{\chi_i(e)}_{\dim \rho_i}$$

$$= \dim \rho_i .$$

Rmk: $\ker \rho_{reg} = \{e\}$.

Pf: $g \cdot h = h \ \forall h$, then $g = e$.

Assume H is a normal subgroup of G.

G/H is the quotient group.

$$\rho : G/H \to GL(V)$$

induces $\tilde{\rho} : G \to G/H \to GL(V)$

Prop: $H = \bigcap \ker \tilde{\rho}_i$

ρ_i irreducible rep'n of G/H.

or $H = \bigcap \ker \tilde{\rho}_i$.

$H \subset \ker \tilde{\rho}_i$

$\tilde{\rho}_i$ irreducible rep'n of G.

Pf: $H \subset \ker \tilde{\rho}_i$ so $H \subset \bigcap_i \ker \tilde{\rho}_i$.

$\bigcap \ker \rho_i = \{eH\}$

ρ_i irreducible rep'ns of G/H

So $\bigcap \ker \tilde{\rho}_i = H$.

---

commutator group $[G, G] = G'$

Defn: $G' = \bigcap H$

H normal subgroup of G.

$H \ni [g, h] \ \forall\, g, h \in G$.

$[g, h] = g h g^{-1} h^{-1}$.

Prop: G/G' is commutative.

Prop: If H normal subgroup and G/H is commutative, then $H \supset [G, G]$

Thm (HW) $G' = \bigcap \ker \chi_i$

$\dim \chi_i = 1$.

dual, tensor product, Hom.

Defn: $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$

$= \{ f : V \to \mathbb{C} \mid f \text{ linear} \}.$

V^* has a G-rep'n structure by

$$(g \cdot f)(v) = f(g^{-1} \cdot v).$$

In terms of matrix rep'n. Let

$B = \{e_1 \dots e_n\}$ be basis of V.

$B' = \{e_1^* \dots e_n^*\}$ be dual basis of V^*

defined by $e_i^*(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Then $$R_{B'}(g) = \left[(R_B(g))^T \right]^{-1}.$$

If V is unitary and $e_1 \dots e_n$ are orthonormal basis, then $(R_B(g)^T)^{-1} = \overline{R_B(g)}$. so $\boxed{\chi_{V^*}(g) = \overline{\chi_V(g)}}$

Tensor product. V, W are G-rep'n's.

$V \otimes W$ has a G-rep'n structure

by $g \cdot (v \otimes w) = (gv) \otimes (gw)$

Ex: V has basis $\{v_1, v_2, v_3\} = \mathcal{B}$

W has basis $\{w_1, w_2\} = \mathcal{C}$.

$V \otimes W$ has basis

$\{v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_1,$
$v_3 \otimes w_3\} = \mathcal{D}$.

$$g \cdot (v_1, v_2, v_3) = (v_1, v_2, v_3) \cdot \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{=A}$$

$$g(w_1, w_2) = (w_1, w_2) \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B.$$

Then $g \cdot D = D \cdot \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{bmatrix}$.

Prop:

$$\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$$

The product of two characters is still a character, but usually not irreducible. if V, W are both irreducible.

Important to know how to decompose $V \otimes W$ into irreducible repn's.

(Clesch-Gordon coefficients in quantum mechanism)

Hom $\quad V, \quad W \quad G$-repn.

Defn: $\mathrm{Hom}_{\mathbb{C}}(V, W) = \{ F : V \to W$ linear transformations $\}$.

$$(g \cdot F)(v) = gF(g^{-1}v)$$

Prop: $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$.

as G-repn's.

Pf: $V^* \otimes W \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, W)$

$f \otimes w \longmapsto (F : V \mapsto f(v) \otimes w)$.

Schur's lemma.

Defn: (G - invariant linear transformations
or G - compatible)

$$\mathrm{Hom}_G(V, W) = \{ F : V \to W \text{ linear trans.} \mid F(g \cdot v) = g \cdot F(v),\ \forall g \in G \}$$

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "g\cdot"'] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, "g\cdot"] \\ V \arrow[r, "F"] & W \end{tikzcd}$$

The diagram commutes.

Prop: $F \in \mathrm{Hom}_G(V, W)$.

$\ker F$ is G-invariant subspace.

$\mathrm{Im}\, F$ is G-invariant subspace.

Check this from definition.

(Schur's lemma) V, W irreducible.

① $\mathrm{Hom}_G(V, W) = 0$ if $V \not\cong W$.

② $\mathrm{Hom}_G(V, V) \cong \mathbb{C}$.

i.e. if $F \in \mathrm{Hom}_G(V, V)$

F is scalar multiplication

$F(v) = c \cdot v$ for some $c \in \mathbb{C}$.

pf: ① If $F \in \mathrm{Hom}_G(V, W)$.

$\ker F = \{0\}$ or V.

If $\ker F = V$, then $F = 0$

If $\ker F = \{0\}$, then $V \hookrightarrow W$

$\mathrm{Im} F \neq \{0\}$, so $\mathrm{Im} F = W$,

$F : V \xrightarrow{\cong} W$

② Let $F \in \mathrm{Hom}_G(V, V)$.

F has eigenvalue c.

eigenspace $\ker(F - c\,\mathrm{Id})$ is G-invariant, and $\neq \{0\}$

so $\ker(F - c\,\mathrm{Id}) = V$.

so $F(v) = c \cdot v$.

Application: Center of G from character table.

Defn: $Z(G) = \{ g \in G \mid gh = hg \ \forall h \in G \}$.

Defn: $Z_\chi = \{ g \in G \mid |\chi(g)| = \chi(e) \}$

$= \{ g \in G \mid \underbrace{\rho(g) = c \cdot \mathrm{Id}}_{c = e^{\frac{2\pi i}{k}}} \}$

from HW.

Prop: $Z(G) = \bigcap_{\chi \text{ irreducible.}} Z_\chi$

Pf: " $\subset$ " If $g \in Z(G)$, ρ irred. repn.

$\rho(g) \cdot \rho(h) = \rho(h) \rho(g) \quad \forall h \in G.$

so $\rho(g) \in \mathrm{Hom}_G(V, V)$

so $\rho(g) = c \cdot Id$, (Schur)

$g \in Z_\chi$.

" $\supset$ " If $g \in \bigcap Z_\chi$,

$$\rho(ghg^{-1}h^{-1}) = \rho(g) \cdot \rho(h) \rho(g)^{-1} \rho(h)^{-1}$$

$$= (c\, Id)\, \rho(h) \cdot (c^{-1} Id)\, \rho(h)^{-1}$$

$$= Id.$$

so $ghg^{-1}h^{-1} \in \ker \rho$, for all ρ irred.

so $ghg^{-1}h^{-1} = e$. $\square$

Proof of orthogonal relations.

Defn: V G-rep'n. The invariant space

is $\mathrm{Inv}_G(V) = \{ v \in V \mid g \cdot v = v,\ \forall g \in G \}$.

Lemma: $$\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Inv}_G(V)}(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \dim \mathrm{Inv}_G(V)$$

$$= \dim \mathrm{Inv}_G(V).$$

Pf: $z = \frac{1}{|G|} \sum_{g \in G} \rho(g)$.

Then $z \cdot \rho(h) = z$ for all $h \in G$.

and $\rho(h) \cdot z = z$

$$z \cdot \rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\rho(h)$$

$$= \frac{1}{|G|} \sum_{g \in G} \rho(gh) = z$$

so $z^2 = z$

z has two eigenspaces.

$\lambda = 1 \quad V_1$

$\lambda = 0 \quad V_0$

$V = V_1 \oplus V_0$. because

$$v = \underbrace{(Id - z)(v)}_{\in V_0} + \underbrace{z(v)}_{\in V_1}.$$

Claim: $V_1 = Inv_G(V)$

"$\subset$" if $v \in V_1$, $z(v) = v$.

then $g \cdot v = \rho(g)(z(v))$

$= z(v) = v$.

"$\supset$" if $g \cdot v = v$, $\forall g \in G$.

then $z(v) = v$.

So $\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \text{Trace } z$

$= \dim V_1$

$= \dim \mathrm{Inv}_G(V)$.

Fact: $\mathrm{Inv}_G(\mathrm{Hom}_G(V, W))$

$= \mathrm{Hom}_G(V, W)$.

Pf: Plug in the defn of

G-rep'n structure on

$\mathrm{Hom}_{\mathbb{C}}(V, W)$

Pf of orthogonal relation.

compute $\frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g)$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{V^* \otimes W}(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot \chi(g)$$

Lemma $= \dim \mathrm{Inv}_G(\mathrm{Hom}_{\mathbb{C}}(V, W))$

$$= \dim \operatorname{Hom}_G(V, W)$$

If V, W irreducible $= \begin{cases} 0 & V \not\cong W \\ 1 & V \cong W. \end{cases}$